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Proper-time expansion of the one-loop effective Lagrangian in powers of derivatives

Josef A Zuk

Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

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Abstract. It is shown that the proper-time method can be adapted to make the calculation of higher derivative terms in the one-loop effective Lagrangian easy.

1. Introduction

The background field method (DeWitt 1965, 1975, Honerkamp 1972, Abbott 1981, Vilkovisky 1984) provides a convenient formalism for studying the loop expansion of the Green functions and the effective action, Γ, of quantum field theory. In particular, it allows direct coordinate space calculations to be done, and hence is suitable for deriving an expansion of the one-loop effective action, $\Gamma^{(1)}$, in powers of derivatives of the fields.

Such a calculation is made possible by using a representation of the one-loop effective action in terms of Schwinger's proper-time Green function (Schwinger 1951) expanded as an asymptotic series in the proper-time parameter, valid for small space-time separation and related to the usual WKB form of DeWitt (DeWitt 1965, 1975). The ultraviolet regularisation of the theory is achieved by dimensional continuation, introduced in this context by Brown (1977). Zeta function regularisation (Birrell and Davies 1982) would be equally appropriate here.

Although the proper-time method has been employed extensively (Bunch 1979, Lee 1982) to find renormalisation counter-terms, especially in curved spacetime, it seems that it has only been used to extract formal expressions for the effective action and no explicit evaluations are in evidence in the literature.

In this paper we consider the case of a self-interacting scalar field, for which we compute the one-loop effective Lagrangian up to fourth order in powers of derivatives of the field. Even though our results will be true more generally, we shall illustrate the procedure on the ϕ^4 model for definiteness, and in order to compare with the expression recently obtained by Fraser (Fraser 1984, Aitchison and Fraser 1984) where an abstract operator expansion was used. As we shall see, one of the main advantages of the present method is that several 'short-cuts' are available which greatly simplify the calculation in comparison with other approaches. In fact, the principal motivation for this work was the simplicity that the method affords which may be of especial value in the treatment of more complicated field theories such as the nonlinear σ model.

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2. Review

Let $S(\phi)$ denote the classical action for a one-component scalar field theory. Then, in the background field method, the (bare) effective action, $\Gamma(\phi)$, is given to be the solution of the equation (Vilkovisky 1984)

$$\exp\left(\frac{\mathrm{i}}{\hbar}\Gamma(\phi)\right)=\int \mathscr{D}\varphi \exp\left[\frac{\mathrm{i}}{\hbar}\left(S(\varphi)+(\phi-\varphi)\frac{\partial\Gamma(\phi)}{\partial\phi}\right)\right] \tag{1}$$

from which the effective Lagrangian, L_{eff}, is defined by

$$\Gamma(\phi)=\int \mathrm{d}x\, L_{\mathrm{eff}}. \tag{2}$$

In the one-loop approximation

$$\Gamma(\phi)=S(\phi)+\frac{\mathrm{i}\hbar}{2}\operatorname{Tr}\ln\left(\frac{\delta^2 S(\phi)}{\delta\phi\,\delta\phi}\right)+\mathrm{O}(\hbar^2). \tag{3}$$

Setting $\hbar=1$, we have for the ϕ^4 model, which has classical Lagrangian

$$L_{\mathrm{cl}}=\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi-\tfrac{1}{2}m^2\phi^2-(\lambda/4!)\phi^4, \tag{4}$$

the one-loop contribution to $\Gamma(\phi)$,

$$\begin{aligned}\Gamma^{(1)}(\phi)&=\tfrac{1}{2}\mathrm{i}\operatorname{Tr}\ln(\partial^2+m^2+\tfrac{1}{2}\lambda\phi^2)\\&=\tfrac{1}{2}\mathrm{i}\ln\det(\partial^2+m^2+\tfrac{1}{2}\lambda\phi^2).\end{aligned} \tag{5}$$

For future reference we define the following quantities:

$$V\equiv\tfrac{1}{2}\lambda\phi^2 \tag{6a}$$

$$M^2\equiv m^2+V \tag{6b}$$

$$W\equiv M^2. \tag{6c}$$

We begin by reviewing the standard Schwinger–DeWitt derivation of $\Gamma^{(1)}(\phi)$ as a series in inverse powers of the mass squared, m^2.

To give precise meaning to the formal expression (5) we introduce the proper-time representation. With spacetime dimension extended to n, it is well known that (Lee 1982)

$$\left\langle x\left|\frac{1}{(-\partial^2-m^2-\frac{1}{2}\lambda\phi^2+\mathrm{i}\varepsilon)}\right|y\right\rangle=-\int_0^\infty \mathrm{i}\,\mathrm{d}s\exp(-s\varepsilon)\langle x,s|y\rangle \tag{7}$$

$$\ln\det(\partial^2+m^2+\tfrac{1}{2}\lambda\phi^2-\mathrm{i}\varepsilon)=-\int_0^\infty\frac{\mathrm{i}\,\mathrm{d}s}{\mathrm{i}s}\exp(-s\varepsilon)\int \mathrm{d}^n x\langle x,s|x\rangle \tag{8}$$

where $\langle x,s|y\rangle$ denotes the proper-time Green function (Schwinger 1951):

$$\langle x,s|y\rangle=\langle x|\exp[-\mathrm{i}s(\partial^2+m^2+\tfrac{1}{2}\lambda\phi^2)]|y\rangle. \tag{9}$$

We have ensured Feynmann boundary conditions by giving the mass parameter, m, an infinitesimal negative imaginary part, $-\mathrm{i}\varepsilon$. This renders the operator $\partial^2+m^2+\frac{1}{2}\lambda\phi^2$ non-singular so that its inverse is well defined. The central idea behind all this is that $\langle x,s|y\rangle$ satisfies the 'Schrödinger equation':

$$-\frac{\partial}{\partial(\mathrm{i}s)}\langle x,s|y\rangle=(\partial_x^2+m^2+\tfrac{1}{2}\lambda\phi^2(x))\langle x,s|y\rangle \tag{10}$$

with boundary condition

$$\lim_{s\to 0^+} \langle x, s|y\rangle = \langle x|y\rangle = \delta^{(n)}(x-y). \tag{11}$$

Now, in calculating $L^{(1)}_{\text{eff}}$, we are interested in the short distance behaviour of the Green function $\langle x, s|y\rangle$; hence, it suffices to express it in the WKB form (DeWitt 1975)

$$\langle x, s|y\rangle = \frac{\mathrm{i}}{(4\pi \mathrm{i}s)^{n/2}} \exp\left(\frac{(x-y)^2}{4\mathrm{i}s} - \mathrm{i}m^2 s\right) F(x, y; \mathrm{i}s; n). \tag{12}$$

Accordingly,

$$L^{(1)}_{\text{eff}} = \frac{1}{2(4\pi)^{n/2}} \int_0^\infty \frac{\mathrm{i}\,\mathrm{d}s}{(\mathrm{i}s)^{1+n/2}} \exp(-s\varepsilon) \exp(-\mathrm{i}m^2 s) F(x, x; \mathrm{i}s; n). \tag{13}$$

Convergence at the upper limit of the s integration is guaranteed by the $-\mathrm{i}\varepsilon$ prescription, but $L^{(1)}_{\text{eff}}$ diverges at the lower end at which $s=0$. If we allow n to be continued analytically throughout the complex plane, then this divergence will appear as a pole singularity as n approaches the relevant physical spacetime dimension; in our case, four. To display this, we expand $F(x, y; \mathrm{i}s; n)$ in a power series about $s=0$,

$$F(x, y; \mathrm{i}s; n) = \sum_{k=0}^{\infty} a_k(x, y)(\mathrm{i}s)^k \tag{14}$$

where the $a_k(x, y)$ are supposed well behaved in the neighbourhood $x=y$. After substituting equation (14) into equation (13) we evaluate the s integral by rotating the contour into the negative imaginary axis, which is equivalent to the change of variable $s=-\mathrm{i}\tau$. In other words, using the formula

$$\int_0^\infty \mathrm{i}\,\mathrm{d}s \exp(-s\varepsilon)(\mathrm{i}s)^{k-1} \exp(-\mathrm{i}m^2 s) = \int_0^\infty \mathrm{d}\tau\, \tau^{k-1} \exp(-m^2\tau) = \Gamma(k)/m^{2k} \tag{15}$$

we find (Bunch 1979, Birrell and Davies 1982)

$$L^{(1)}_{\text{eff}} = \frac{1}{2(4\pi)^{n/2}} \left(\frac{m}{\mu}\right)^{n-4} \sum_{k=0}^{\infty} a_k(x, x) m^{4-2k} \Gamma(k-n/2) \tag{16}$$

where we have introduced an auxiliary mass parameter, μ, in order to keep the dimension of $L^{(1)}_{\text{eff}}$ fixed as $(\text{length})^{-4}$ for arbitrary spacetime dimension, n.

It is clear that if $n=4$, there are Γ-function poles in the terms corresponding to $k=0$, 1 and 2. Making the following expansions:

$$\left(\frac{m}{\mu}\right)^{n-4} = 1 + \tfrac{1}{2}(n-4)\ln\frac{m^2}{\mu^2} + \mathrm{O}((n-4)^2) \tag{17a}$$

$$\Gamma(2-n/2) = \frac{2}{4-n} - \gamma + \mathrm{O}((n-4)) \tag{17b}$$

$$\Gamma(1-n/2) = \frac{2}{2-n}\left(\frac{2}{4-n} - \gamma\right) + \mathrm{O}((n-4)) \tag{17c}$$

$$\Gamma(-n/2) = \frac{4}{n(n-2)}\left(\frac{2}{4-n} - \gamma\right) + \mathrm{O}((n-4)) \tag{17d}$$

where γ is the Euler constant, we can decompose $L^{(1)}_{\text{eff}}$ into divergent and finite parts according to

$$L_{\mathrm{eff}}^{(1)} = L_{\mathrm{div}}^{(1)} + L_{\mathrm{fc}}^{(1)} + L_{\mathrm{ren}}^{(1)} \tag{18}$$

where the pole term

$$L_{\mathrm{div}}^{(1)} = -\frac{1}{(4\pi)^{n/2}}\frac{1}{n-4}\left(\frac{4m^4 a_0}{n(n-2)} - \frac{2m^2 a_1}{n-2} + a_2\right) \tag{19}$$

is removed by introducing appropriate renormalisation counter-terms, and in the limit $n \to 4$ the finite counter-term, $L_{\mathrm{fc}}^{(1)}$, and the renormalised effective Lagrangian, $L_{\mathrm{ren}}^{(1)}$, are given by

$$L_{\mathrm{fc}}^{(1)} = -\frac{1}{2(4\pi)^2}\left(\gamma + \ln\frac{m^2}{\mu^2}\right)\left(\tfrac{1}{2}m^4 a_0 - m^2 a_1 + a_2\right) \tag{20}$$

$$L_{\mathrm{ren}}^{(1)} = \frac{1}{2(4\pi)^2}\sum_{k=3}^{\infty} a_k(x,x) m^{4-2k}\Gamma(k-2). \tag{21}$$

The value of the auxiliary mass parameter, μ, is arbitrary and any change in it is precisely compensated by a change in the renormalisation counter-terms.

3. The coefficient functions

Next, we outline the strategy for determining the a_k. Inserting equation (12) into the 'Schrödinger equation' (10) we find (Lee 1982)

$$-\frac{\partial}{\partial(\mathrm{i}s)}F(x,y;\mathrm{i}s;n) = \left(\partial_x^2 + V(x) + \frac{(x-y)_\mu}{\mathrm{i}s}\partial_x^\mu\right)F(x,y;\mathrm{i}s;n) \tag{22}$$

which in turn leads to a recursive system of differential equations for the coefficient functions $a_k(x,y)$,

$$-(x-y)^\mu \partial_\mu a_0 = 0 \tag{23a}$$

$$-k a_k = (x-y)^\mu \partial_\mu a_k + (\partial^2 + V)a_{k-1} \tag{23b}$$

for $k = 1, 2, \ldots$. We recall that $V(x) \equiv \frac{1}{2}\lambda\phi^2(x)$. From the boundary condition $a_0(x,x) = 1$ it follows immediately that $a_0(x,y) = 1$.

Only the coincidence limits of the a_k are relevant to us. These can be obtained as local functions of the field ϕ by successive differentiations of the recurrence equations (23). Let us define

$$\bar{a}_k(x) \equiv a_k(x,x) \tag{24}$$

and

$$\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}\bar{a}_k \equiv \lim_{y\to x}\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}a_k(x,y) \tag{25}$$

where the derivatives are all taken with respect to the variable x. Successive application of Leibnitz' rule to equations (23) yields the set of relations

$$\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}\bar{a}_k = -\frac{1}{k+N}\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}(\partial^2 + V)\bar{a}_{k-1} \tag{26}$$

for $k=1,2,\ldots$ and $N=0,1,2,\ldots$. By repeated use of the above relations and the fact that $a_0(x,y)=1$, explicit solutions for the $\bar{a}_k$ may be obtained. As an example we compute $\bar{a}_2$:

$$\begin{aligned}\bar{a}_2 &= -\tfrac{1}{2}(\partial^2+V)\bar{a}_1\\ &= \frac{1}{3\times 2}\partial^2(\partial^2+V)\bar{a}_0+\tfrac{1}{2}V(\partial^2+V)\bar{a}_0\\ &= \tfrac{1}{6}\partial^2 V+\tfrac{1}{2}V^2.\end{aligned}$$

It should now be apparent that the series (16) is not suitable for extracting an expansion in powers of derivatives because any given order in the derivatives appears in an infinite number of the terms $\bar{a}_k$. To remedy this, we make the important observation that $L_{\text{eff}}^{(1)}$ should depend only on M^2 and its derivatives. This suggests that an expansion in inverse powers of M^2 rather than m^2 is called for.

4. Expansion in powers of derivatives

In view of the preceding discussion, we rewrite $L_{\text{eff}}^{(1)}$ in the form

$$L_{\text{eff}}^{(1)}=\frac{1}{2(4\pi)^{n/2}}\int_0^\infty \frac{\mathrm{i}ds}{(\mathrm{i}s)^{1+n/2}}\exp(-s\varepsilon)\exp(-\mathrm{i}M^2(x)s)\hat{F}(x,x;\mathrm{i}s;n) \quad (27)$$

$$\hat{F}(x,y;\mathrm{i}s;n)=\sum_{k=0}^{\infty}\hat{a}_k(x,y)(\mathrm{i}s)^k. \quad (28)$$

The boundary condition at $s=0$ is still $\hat{F}(x,x;0;n)=1$. Following the same reasoning as in § 2 we find that the pole part, $L_{\text{div}}^{(1)}$, of $L_{\text{eff}}^{(1)}$ and its finite contributions, namely $L_{\text{fc}}^{(1)}$ and $L_{\text{ren}}^{(1)}$, are now given by

$$L_{\text{div}}^{(1)}=-\frac{1}{(4\pi)^{n/2}(n-4)}\left(\frac{4M^4\hat{a}_0}{n(n-2)}-\frac{2M^2\hat{a}_1}{n-2}+\hat{a}_2\right) \quad (29)$$

$$L_{\text{fc}}^{(1)}=-\frac{1}{2(4\pi)^2}(\gamma-\ln\mu^2)(\tfrac{1}{2}M^4\hat{a}_0-M^2\hat{a}_1+\hat{a}_2) \quad (30)$$

and

$$\begin{aligned}L_{\text{ren}}^{(1)}=&-\frac{1}{2(4\pi)^2}\ln M^2(\tfrac{1}{2}M^4\hat{a}_0-M^2\hat{a}_1+\hat{a}_2)\\ &+\frac{1}{2(4\pi)^2}\left(\frac{1}{M^2}\hat{a}_3+\frac{1}{M^4}\hat{a}_4+\frac{2}{M^6}\hat{a}_5+\frac{6}{M^8}\hat{a}_6+\ldots\right).\end{aligned} \quad (31)$$

Now, although one may derive a set of recursion relations for the new coefficient functions $\hat{a}_k$ in a manner analogous to that for the original a_k, they turn out to be very complicated and indeed intractable. A more profitable approach is to realise that

$$\hat{F}(x,y;\mathrm{i}s;n)=\exp(\mathrm{i}V(x))F(x,y;\mathrm{i}s;n) \quad (32)$$

leading to the following expression for the a_k in terms of the $\hat{a}_k$ which satisfy a simple system of equations:

$$\hat{a}_k = \sum_{j=0}^{k} \frac{V^{k-j}}{(k-j)!} a_j. \tag{33}$$

The first three coincidence limits are easily found to be

$$\hat{\bar{a}}_0 = 1 \tag{34a}$$

$$\hat{\bar{a}}_1 = 0 \tag{34b}$$

$$\hat{\bar{a}}_2 = \tfrac{1}{6}\partial^2 V. \tag{34c}$$

Before proceeding further we note that $L^{(1)}_{\text{div}}$ has not been altered by the resummation, i.e.

$$\begin{aligned}\tfrac{1}{2}M^4\hat{\bar{a}}_0 - M^2\hat{\bar{a}}_1 + \hat{\bar{a}}_2 &= \tfrac{1}{2}m^4\bar{a}_0 - m^2\bar{a}_1 + \bar{a}_2 \\ &= \tfrac{1}{2}(m^2 + V)^2 + \tfrac{1}{6}\partial^2 V.\end{aligned} \tag{35}$$

This means that no problems arise in finding renormalisation counter-terms to absorb the singularity.

To facilitate ease of calculation of the higher coefficient functions we shall develop a few 'short-cut' rules; but first we need to explain some 'book-keeping'. Let us assign a 'weight' $w = +1$ and a 'dimension' $d = -1$ to each 'V', and $w = +1$, $d = +1$ to each derivative pair '$\partial_\mu \partial^\mu$' which appear in the terms arising from the reduction of the coincidence limits $\bar{a}_k$ by applying equations (26). Clearly, the coefficient function $\bar{a}_k$ is composed of terms, made up of V and ∂, each of which has total 'weight' $w = k$ and whose total 'dimensions' range from a minimum $d = -w$ to a maximum $d = w - 2$, i.e.

$$w(\bar{a}_k) = k \qquad d_{\min}(\bar{a}_k) = -w \qquad d_{\max}(\bar{a}_k) = w - 2. \tag{36}$$

Thus we may also assign weights and dimensions to 'composite' terms, for example $\partial_\mu V \partial^\mu \partial^2(V\bar{a}_2)$, which has $w = 6$, $d_{\min} = -2$ and $d_{\max} = 0$. Now, the number of derivative pairs appearing in a term is given by

$$\#\partial^2 = \tfrac{1}{2}(w + d) \tag{37}$$

from which it follows that

$$\#\partial^2_{\min}(\bar{a}_k) = 0 \qquad \#\partial^2_{\max}(\bar{a}_k) = k - 1. \tag{38}$$

However, the situation is improved if we consider the $\hat{\bar{a}}_k$. It is not difficult to convince oneself that

$$\#\partial^2_{\min}(\hat{\bar{a}}_k) = \min\{\chi \in \mathbb{N}: \chi \geqslant k/3\} \qquad \text{for } k = 2, 3, \ldots$$

with

$$\#\partial^2_{\min}(\hat{\bar{a}}_0) = \#\partial^2_{\min}(\hat{\bar{a}}_1) = 0 \tag{39}$$

and

$$\#\partial^2_{\max}(\hat{\bar{a}}_k) = k - 1 \qquad \text{for } k = 1, 2, \ldots. \tag{40}$$

Consequently, contributions to the terms in $L^{(1)}_{\text{ren}}$ quadratic in derivatives arise solely from $\hat{\bar{a}}_2$ and $\hat{\bar{a}}_3$, while the quartic contributions come from $\hat{\bar{a}}_3$, $\hat{\bar{a}}_4$, $\hat{\bar{a}}_5$ and $\hat{\bar{a}}_6$. We also note that

$$w(\hat{\bar{a}}_k) = k \qquad d_{\max}(\hat{\bar{a}}_k) = k - 2 \qquad d_{\min}(\hat{\bar{a}}_k) = 2\#\partial^2_{\min}(\hat{\bar{a}}_k) - k. \tag{41}$$

We now proceed to state the 'short-cut' rules. In the process of applying equations (26) to evaluate $\hat{\bar{a}}_k$:

(i) discard any terms which appear with more derivatives than the desired order;

(ii) discard any terms with $d_{\max} < d_{\min}(\hat{\bar{a}}_k)$ since these must all cancel out in the end; and finally,

(iii) since $L_{\text{eff}}^{(1)}$ depends only on $W(x) = m^2 + V(x)$ and its derivatives, all terms in which V appears undifferentiated must eventually cancel out because the recurrence equations do not involve the mass, m, nor does the expansion (31) explicitly. These can be discarded immediately. (We note, in particular, that the expression for $\hat{\bar{a}}_k$ emerges entirely from the reduction of $\bar{a}_k$.)

5. Results

We obtain the following results which we write out up to terms quartic in the derivatives. (For the purpose of illustration, calculation of the most complicated case, $\hat{\bar{a}}_6$, is presented in the appendix.)

$$\hat{\bar{a}}_0 = 1 \qquad \hat{\bar{a}}_1 = 0 \qquad \hat{\bar{a}}_2 = \tfrac{1}{6}\partial^2 W \tag{42a, b, c}$$

$$\hat{\bar{a}}_3 = -\tfrac{1}{12}\partial_\mu W \partial^\mu W - \tfrac{1}{60}\partial^2\partial^2 W \tag{42d}$$

$$\hat{\bar{a}}_4 = \tfrac{1}{72}\partial^2 W \partial^2 W + \tfrac{1}{30}\partial_\mu W \partial^\mu \partial^2 W + \tfrac{1}{90}\partial_\mu \partial_\nu W \partial^\mu \partial^\nu W + \ldots \tag{42e}$$

$$\hat{\bar{a}}_5 = -\tfrac{1}{72}\partial^2 W \partial_\mu W \partial^\mu W - \tfrac{1}{60}\partial_\mu W \partial_\nu W \partial^\mu \partial^\nu W + \ldots \tag{42f}$$

$$\hat{\bar{a}}_6 = \tfrac{1}{288}(\partial_\mu W \partial^\mu W)^2 + \ldots . \tag{42g}$$

Recall that $W \equiv M^2$. Insertion of these coefficient functions into equation (31) followed by some integration by parts leads to our final expression for $L_{\text{ren}}^{(1)}$:

$$\begin{aligned} L_{\text{ren}}^{(1)} = \frac{1}{2(4\pi)^2}\Bigg(& -\frac{1}{2}W^2 \ln W + \frac{1}{12}\frac{\partial_\mu W \partial^\mu W}{W} \\ & + \frac{1}{120}\frac{\partial^2 W \partial^2 W}{W^2} + \frac{1}{45}\frac{\partial_\mu W \partial_\nu W \partial^\mu \partial^\nu W}{W^3} - \frac{7}{240}\frac{(\partial_\mu W \partial^\mu W)^2}{W^4}\Bigg) + \mathrm{O}(\partial^6). \end{aligned} \tag{43}$$

It is now a straightforward task to compare with the result obtained by Fraser (1984) for the ϕ^4 model. Making the replacement $W = m^2 + \frac{1}{2}\lambda\phi^2$ and performing some further integrations by parts we obtain

$$\frac{\partial_\mu W \partial^\mu W}{W} = \lambda^2 \frac{\phi^2 \partial_\mu \phi \partial^\mu \phi}{(m^2 + \frac{1}{2}\lambda\phi^2)} \tag{44a}$$

$$\frac{(\partial_\mu W \partial^\mu W)^2}{W^4} = \lambda^4 \frac{\phi^4 (\partial_\mu \phi \partial^\mu \phi)^2}{(m^2 + \frac{1}{2}\lambda\phi^2)^4} \tag{44b}$$

$$\begin{aligned} \frac{\partial^2 W \partial^2 W}{W^2} \simeq & -\lambda^2 \frac{(\partial_\mu \phi \partial^\mu \phi)^2}{(m^2 + \frac{1}{2}\lambda\phi^2)^2} + \lambda^2 \frac{\phi^2 (\Box \phi)^2}{(m^2 + \frac{1}{2}\lambda\phi^2)^2} \\ & - 4\lambda^2 \frac{\phi \partial_\mu \phi \partial_\nu \phi \partial^\mu \partial^\nu \phi}{(m^2 + \frac{1}{2}\lambda\phi^2)^2} + 4\lambda^3 \frac{\phi^2 (\partial_\mu \phi \partial^\mu \phi)^2}{(m^2 + \frac{1}{2}\lambda\phi^2)^3} \end{aligned} \tag{44c}$$

$$\frac{\partial_\mu W \partial_\nu W \partial^\mu \partial^\nu W}{W^3} \simeq \lambda^3 \frac{\phi^3 \partial_\mu \phi \partial_\nu \phi \partial^\mu \partial^\nu \phi}{(m^2 + \frac{1}{2}\lambda\phi^2)^3} + \lambda^3 \frac{\phi^2 (\partial_\mu \phi \partial^\mu \phi)^2}{(m^2 + \frac{1}{2}\lambda\phi^2)^3} \tag{44d}$$

where the symbol '$\simeq$' denotes equality up to a total derivative. We deduce that the contributions to the one-loop effective Lagrangian for ϕ^4 theory, quadratic and quartic in derivatives of ϕ are given respectively by

$$\frac{\lambda^2}{2(4\pi)^2}\frac{\phi^2\partial_\mu\phi\partial^\mu\phi}{12(m^2+\frac{1}{2}\lambda\phi^2)} \tag{45}$$

and

$$\frac{\lambda^2}{2(4\pi)^2}\frac{1}{120(m^2+\frac{1}{2}\lambda\phi^2)^2}\bigg[\bigg(-1+\frac{20\lambda\phi^2}{3(m^2+\frac{1}{2}\lambda\phi^2)}-\frac{7\lambda^2\phi^4}{2(m^2+\frac{1}{2}\lambda\phi^2)^2}\bigg)(\partial\phi)^4 + \phi^2(\Box\phi)^2-4\bigg(1-\frac{2\lambda\phi^2}{3(m^2+\frac{1}{2}\lambda\phi^2)}\bigg)\phi\partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi\bigg] \tag{46}$$

in agreement with Fraser. In conclusion, we mention that our result is applicable to any renormalisable self-interacting scalar field theory with a positive, one component Hessian, $W=-\partial^2L/\partial\phi\partial\phi$, where $L(\phi,\partial\phi)$ is the classical Lagrangian.

6. Epilogue: an alternative viewpoint

Suppose we want to evaluate the effective Lagrangian at a particular spacetime point, x_0. Then let us define

$$\tilde{W}(x)\equiv W(x)-W(x_0) \tag{47a}$$

and let us denote

$$W_0\equiv W(x_0). \tag{47b}$$

An expression for $L^{(1)}_{\text{ren}}$ appropriate to this splitting may be obtained by making the replacement $m^2\to W_0$ in equation (16), whence we can take

$$L^{(1)}_{\text{ren}}=\frac{1}{2(4\pi)^2}\bigg(-\ln W_0(\tfrac{1}{2}W_0^2\tilde{\hat{a}}_0-W_0\tilde{\hat{a}}_1+\tilde{\hat{a}}_2)+\frac{1}{W_0}\tilde{\hat{a}}_3+\frac{1}{W_0^2}\tilde{\hat{a}}_4+\dots\bigg). \tag{48}$$

The recursion relations for the corresponding coefficient functions, $\tilde{a}_k$, are then given by equations (23a) and (23b) if we replace $V(x)$ by $\tilde{W}(x)$, i.e.

$$\tilde{a}_k=a_k[\tilde{W}]. \tag{49}$$

But since $\tilde{W}(x_0)=0$, the coincidence limits at $x=x_0$, $\tilde{a}_k(x_0)$, depend only on the derivatives of W and hence will automatically generate the required expansion. Also, since $\partial_\mu W=\partial_\mu\tilde{W}$, and similarly for higher derivatives,

$$\tilde{\hat{a}}_k(x_0)=\hat{a}_k[\tilde{W}](x_0)=\hat{a}_k[W](x_0). \tag{50}$$

Consequently, the computation proceeds in a manner essentially identical to the previous one; only our point of view has changed. In fact, this version may be regarded as a special case of the preceding discussion. Finally, having calculated $L^{(1)}_{\text{ren}}(x_0)$ from equation (48), we recall that the point x_0 was chosen arbitrarily. As before, we obtain expression (43).

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Appendix

$$
\begin{aligned}
\hat{\bar{a}}_6 &\to \bar{a}_6 \\
&\to -\tfrac{1}{6}\partial^2\bar{a}_5 \\
&\to \tfrac{1}{42}\partial^2(\partial^2+V)\bar{a}_4 \\
&\to -\tfrac{1}{336}\partial^2\partial^2(V\bar{a}_3)+\tfrac{1}{42}\partial^2 V\bar{a}_4+\tfrac{1}{21}\partial_\mu V\partial^\mu\bar{a}_4 \\
&\to -\tfrac{1}{336}\partial^2\partial^2(V\bar{a}_3)-\tfrac{1}{168}\partial^2 V\partial^2\bar{a}_3-\tfrac{1}{105}\partial_\mu V\partial^\mu(\partial^2+V)\bar{a}_3 \\
&\to -\tfrac{1}{336}\partial^2\partial^2(V\bar{a}_3)+\tfrac{1}{840}\partial^2 V\partial^2(V\bar{a}_2)+\tfrac{1}{630}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2)+\tfrac{1}{315}\partial_\mu V\partial^\mu V\partial^2\bar{a}_2 \\
&\to -\tfrac{1}{366}\partial^2\partial^2(V\bar{a}_3)+\tfrac{1}{420}\partial^2 V\partial_\mu V\partial^\mu\bar{a}_2+\tfrac{1}{630}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2)-\tfrac{1}{1260}\partial_\mu V\partial^\mu V\partial^2(V\bar{a}_1) \\
&\to -\tfrac{1}{336}\partial^2\partial^2(V\bar{a}_3)+\tfrac{1}{630}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2)-\tfrac{1}{630}\partial_\mu V\partial^\mu V\partial_\nu V\partial^\nu\bar{a}_1 \\
&\to -\tfrac{1}{336}\partial^2\partial^2(V\bar{a}_3)+\tfrac{1}{630}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2)+\tfrac{1}{1260}(\partial_\mu V\partial^\mu V)^2.
\end{aligned}
$$

Now

$$
\begin{aligned}
\partial^2\partial^2(V\bar{a}_3) &\to \partial^2\partial^2 V\bar{a}_3+4\partial_\mu\partial^2 V\partial^\mu\bar{a}_3+2\partial^2 V\partial^2\bar{a}_3+4\partial_\mu\partial_\nu V\partial^\mu\partial^\nu\bar{a}_3+4\partial_\mu V\partial^\mu\partial^2\bar{a}_3 \\
&\to -\partial_\mu\partial^2 V\partial^\mu(V\bar{a}_2)-\tfrac{2}{5}\partial^2 V\partial^2(V\bar{a}_2)-\tfrac{4}{5}\partial_\mu\partial_\nu V\partial^\mu\partial^\nu(V\bar{a}_2)-\tfrac{2}{3}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2) \\
&\to -\tfrac{4}{5}\partial^2 V\partial_\mu V\partial^\mu\bar{a}_2-\tfrac{8}{5}\partial_\mu\partial_\nu V\partial^\mu V\partial^\nu\bar{a}_2-\tfrac{2}{3}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2) \\
&\to -\tfrac{2}{3}\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_\mu V\partial^\mu\partial^2(V\bar{a}_2) &\to \partial_\mu V\partial^\mu\partial^2 V\bar{a}_2+\partial^2 V\partial_\mu V\partial^\mu\bar{a}_2+2\partial_\mu V\partial^\mu\partial^\nu V\partial_\nu\bar{a}_2 \\
&\qquad +2\partial_\mu V\partial_\nu V\partial^\mu\partial^\nu\bar{a}_2+\partial_\mu V\partial^\mu V\partial^2\bar{a}_2 \\
&\to -\tfrac{1}{2}\partial_\mu V\partial_\nu V\partial^\mu\partial^\nu(V\bar{a}_1)-\tfrac{1}{4}\partial_\mu V\partial^\mu V\partial^2(V\bar{a}_1) \\
&\to -\partial_\mu V\partial_\nu V\partial^\mu V\partial^\nu a_1-\tfrac{1}{2}\partial_\mu V\partial^\mu V\partial_\nu V\partial^\nu\bar{a}_1 \\
&\to \tfrac{3}{4}(\partial_\mu V\partial^\mu V)^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\hat{\bar{a}}_6 &= \left(\tfrac{1}{1260}+\tfrac{3}{4}\,\tfrac{1}{630}+\tfrac{3}{4}\,\tfrac{2}{3}\,\tfrac{1}{336}\right)(\partial_\mu V\partial^\mu V)^2+\ldots \\
&= \tfrac{1}{288}(\partial V)^4+\ldots.
\end{aligned}
$$

One should note that the $\hat{\bar{a}}_6$ term corresponds, in perturbation theory, to the box graph (figure A1).

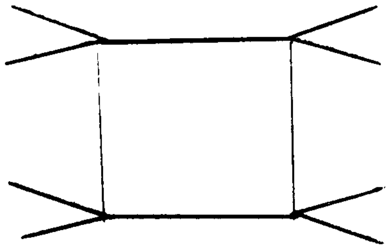

Figure A1.

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